Eur. Phys. J. D **29**, 379–389 (2004)
DOI: 10.1140/epjd/e2004-00036-0

THE EUROPEAN PHYSICAL JOURNAL D

Fokker-Planck modeling for particle slowing down and thermalization in a Maxwellian plasma

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Received 25 July 2003 / Received in final form 1st December 2003
Published online 16 March 2004 –

Abstract. In this paper the Fokker-Planck equation, which takes into account the medium absorption and the Maxwellian behavior of the field particles at low energy for Coulomb interactions is obtained. The analytical solution of the stationary distribution function is obtained in both angle and velocity variables. In particular the electron distribution for electron-ion collisions has been obtained using this diffusion approximation for beam and isotropic sources. If the absorption is neglected the solution recovers the classical stationary Maxwell distribution. For low absorption rates the solution shows a typical slowing down spectrum for high energy and a Maxwellian-like distribution at thermal energy. For moderate and high absorption rates the test particles do not reach the thermal equilibrium and the Maxwell distribution at low energies does not appear.

PACS. 51.10.+y Kinetic and transport theory of gases – 41.75.-i Charged-particle beams – 52.65.Ff Fokker-Planck and Vlasov equation

1 Introduction

The collisional slowing down of a beam of high energy charged particles in a background plasma or in a solid sample is a problem of fundamental importance for applications ranging from fusion plasmas to beam microprobe analysis. In fusion plasmas examples of such high energy beams are numerous and appear in the form of fusion generated alpha particles, neutral beam injected particles, ICRF heated particles and runaway electrons. In the electron microprobe analysis electron beams are used to generate X-rays in the sample to be analyzed. From the wavelength and intensity of the lines in the X-ray spectrum the present elements may be identified and their concentrations estimated. Electron beams generate not only characteristic X-ray lines but also a continuous spectrum (Bremsstrahlung) consisting of photons emitted by electrons suffering deceleration in collisions with atoms. All these applications can be investigated by using the Fokker-Planck equation, which describes the collisional dynamics of the charged particles including the effects of frictional slowing down, energy diffusion and pitch-angle scattering [4–9]. The results presented in this paper refer to the case of a pure Coulomb scattering with Debye shielding; this is the typical case of slowing down in plasmas. However the method herein introduced is similarly valid also for the case of slowing down in solids. In fact the main differences between the two physical systems can be accounted with different cross-sections and different shielding properties. The Coulomb differential cross-section can be substituted for example by Brooks and Herring-type cross-sections which are valid in solid state problems [11]. The Brooks and Herring collisional model presents in fact corrections to the Coulomb cross-section that take into account the effective mass of the field particles in lattices and the effective atomic potential screening (usually in the form of Thomas-Fermi-type screenings). It is therefore immediate and straightforward to apply the method given in this paper also to solid state problems. In literature [1–3, 10] the field particles have always been considered at rest in comparison with the velocity of the test particles and this leads to a complete neglecting of the particle spectrum at low energy. However in many applications the low part of the spectrum is fundamental and should be taken into account. In the present paper we reconsider the slowing down problem and include the influence of the presence of a Maxwellian field completing the analysis over all the energy ranges. In this new form the Fokker-Planck equation becomes a powerful tool for the investigation of processes in plasma or solid state physics. The slowing down problem of test particles colliding with field particles of a given medium has been appropriately studied in the framework of the Fokker-Planck

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equation [13,14,17,19]

$$\frac{\partial f(t,\vec{r},\vec{v})}{\partial t} + \vec{v}\cdot\vec{\nabla}_r f(t,\vec{r},\vec{v}) + \langle \Sigma_a(v)\rangle v f(t,\vec{r},\vec{v}) = - \vec{\nabla}\cdot[\langle \Delta\vec{v}\rangle f(t,\vec{r},\vec{v})] + \frac{1}{2}\vec{\nabla}\vec{\nabla} : [\langle \Delta\vec{v}\Delta\vec{v}\rangle f(t,\vec{r},\vec{v})] + S(t,\vec{r},\vec{v}), \quad (1)$$

where $f(t,\vec{r},\vec{v})$ is the test particle distribution function in μ phase space,

$$\langle \Delta\vec{v}\rangle = \int_{\Re^3}\int_{\Omega_c} \Delta\vec{v} g f_f(\vec{V})\sigma(\vec{\omega},\vec{g})d\vec{\omega}d\vec{V} \quad (2)$$

is the dynamical-friction coefficient and

$$\langle \Delta\vec{v}\Delta\vec{v}\rangle = \int_{\Re^3}\int_{\Omega_c} (\Delta\vec{v}\Delta\vec{v}) g f_f(\vec{V})\sigma(\vec{\omega},\vec{g})d\vec{\omega}d\vec{V} \quad (3)$$

is the diffusion-in-velocity tensor. The function $f_f(\vec{V})$ is the velocity distribution function of the n_f field particles which, in order to extend the validity of the Fokker-Planck equation to the thermal energy range, is supposed to be the stationary Maxwell distribution

$$f_f\left(\vec{V}\right) = M(\vec{V}) = n_f\left(\frac{m_f}{2\pi k T_f}\right)^{\frac{3}{2}} e^{-\frac{m_f V^2}{2kT_f}}. \quad (4)$$

T_f and m_f are the field particle temperature and mass respectively and k is the Boltzmann constant. $\Delta\vec{v}$ and $g = |\vec{v} - \vec{V}|$ are the change in velocity of the test particles and the relative velocity between field and test particles respectively. $S(t,\vec{r},\vec{v})$ is the test particle source. In equations (2, 3) the integration of the scattering center of mass ($C.M.$) angle $\vec{\omega} = (\chi,\phi)$ and of the velocity field $\vec{V}$ must be performed over the unit sphere Ω_c and $\Re^3$ respectively. The absorption rate is determined by the macroscopic absorption cross-section $\Sigma_a(v)$ which is assumed function of particle energy E. At the low level expansion it is possible to take $\Sigma_a(v) = a'/E = a/v^2$ with a non-negative constant [18].

The differential Coulomb cross-section $\sigma(\chi,\vec{g})$ is

$$\sigma(\chi,g)d\vec{\omega}d\phi = \frac{Z'^2 Z^2 e^4}{16\pi^2\epsilon_0^2 m'^2 (1-\cos\chi)^2}\frac{\sin\chi}{g^4}d\chi d\phi, \quad (5)$$

where $m' = m_f m_t/(m_t + m_f)$ and m_t is the test particle mass. This form is the standard elastic scattering cross-section for test and field particles of charge $Z'e$ and Ze respectively.

The scalar product (double-dot product) by two tensors $\widetilde{a}:\widetilde{b}$ is defined by the following expression $\widetilde{a}:\widetilde{b} = \sum_i^3\sum_j^3 a_{ij}b_{ji}$. In particular we are interested in $\vec{\nabla}\vec{\nabla}:\langle\Delta\vec{v}\Delta\vec{v}\rangle$, introduced in (1), which is easily transformed into a divergence of a single dot product as [14]

$$\vec{\nabla}\vec{\nabla} : \langle \Delta\vec{v}\Delta\vec{v}\rangle = \sum_i^3\sum_j^3 \nabla_i\nabla_j\langle\Delta v\Delta v\rangle_{ji} = \vec{\nabla}\cdot\left(\vec{\nabla}\cdot\langle\Delta\vec{v}\Delta\vec{v}\rangle\right).$$

If the current $\vec{J}(t,\vec{r},\vec{v})$ is introduced by

$$\vec{J}(t,\vec{r},\vec{v}) = -[\langle\Delta\vec{v}\rangle f(t,\vec{r},\vec{v})] + \frac{1}{2}\vec{\nabla}\cdot[\langle\Delta\vec{v}\Delta\vec{v}\rangle f(t,\vec{r},\vec{v})], \quad (6)$$

then the Fokker-Planck equation in (1) can be rewritten in the appealing diffusion form

$$\frac{\partial f(t,\vec{r},\vec{v})}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}} f(t,\vec{r},\vec{v}) + \langle\Sigma_a\rangle v f(t,\vec{r},\vec{v}) = -\vec{\nabla}_{\vec{v}}\cdot\vec{J}(t,\vec{r},\vec{v}) + S(t,\vec{r},\vec{v}). \quad (7)$$

The Fokker-Planck system (6, 7) is completed by the appropriate initial and boundary conditions. In this paper, we compute without any approximation the drift coefficient and the diffusion tensor and investigate the physical meaning of this equation. In some limiting cases the Fokker-Planck diffusion equation is able to reproduce the Boltzmann equation solution for the same cases. We investigate such limiting cases and solve the stationary one. If the absorption is neglected the solution recovers the classical stationary Maxwell distribution. For low absorption rates the solution shows a typical slowing down spectrum for high energy and a Maxwellian-like distribution at thermal energy (conventionally taken to be 0.025 eV). For moderate and high absorption rates the test particles do not reach the thermal equilibrium and the Maxwell distribution does not appear.

The paper is organized as follows. In Section 2, we compute the Fokker-Planck equation coefficients with the Maxwellian field particle distribution introduced above. The Fokker-Planck equation and its approximation are discussed in Section 3 and finally simple analytical solution is presented in Section 4.

2 Computation of the coefficients with a Maxwellian field particle distribution

In this section, we calculate the Fokker-Planck coefficients for charged test particles colliding with a Maxwellian field particle distribution. These coefficients are in agreement in the corresponding limit with those in literature (see for example [12]). In particular we need to compute the dynamical-friction vector defined by (2) and the diffusion-in-velocity tensor defined by (3). In order to represent the velocity vector of the test particles after and before collisions we use two different coordinates systems: the laboratory coordinate system and the relative coordinate system. In the laboratory system, we use spherical coordinates (v,θ,φ) with initial velocity, v_0, as the polar axis. In order to describe the collision between test and field particles we use the relative coordinate system in Cartesian coordinates (v_1,v_2,v_3) with relative velocity $\vec{g} = \vec{v} - \vec{V}$ along the direction $\widehat{i}_1$. The unit vector $\widehat{i}_2$ can be taken perpendicular to the plane defined by $\widehat{i}_1$, $\vec{v}$ and $\widehat{i}_3 = \widehat{i}_1 \times \widehat{i}_2$. With this representation the velocity change in the laboratory system $\Delta\vec{v}$ and the tensor $\Delta\vec{v}\Delta\vec{v}$ can be determined as a function of $\vec{g}$, χ and ϕ. The velocity change $\Delta\vec{v}$ of a binary elastic collision in the (v_1, v_2, v_3) system is given by

$$\begin{aligned}\Delta\vec{v} &= \Delta v_1\hat{i}_1 + \Delta v_2\hat{i}_2 + \Delta v_3\hat{i}_3 \\ &= -Mg(1-\cos\chi)\hat{i}_1 + Mg\sin\chi\cos\phi\hat{i}_2 \\ &\quad + Mg\sin\chi\sin\phi\hat{i}_3, \end{aligned} \quad (8)$$

where $M = m'/m_t = m_f/(m_t + m_f)$ and χ is the electron change in direction in the *C.M.* system. The angle ϕ is the angle between the direction $\hat{i}_2$ and the projection over the $(\hat{i}_2, \hat{i}_3)$-plane. This means that during the collision the relative velocity $\vec{g} = \vec{v} - \vec{V}$ rotates by $\vec{\omega} = (\chi, \phi)$ around the axis $\hat{i}_1$ keeping the module g constant. We note that

$$\begin{aligned}\Delta v_1\hat{i}_1 &= -Mg(1-\cos\chi)\hat{i}_1 = -M(1-\cos\chi)\vec{g} \\ &= -M(1-\cos\chi)(\vec{v}-\vec{V}). \end{aligned} \quad (9)$$

Clearly $\vec{v} = v\hat{i}_v$ and $\vec{v} = V\cos\zeta\hat{i}_v + V\sin\zeta\cos\eta\hat{i}_\theta + V\sin\zeta\sin\eta\hat{i}_\varphi$ where ζ is the angle between $\vec{v}$ and $\vec{V}$ for which we have $g^2 = v^2 + V^2 - 2vV\cos\zeta$. The angle η is the angle between $\hat{i}_3$ and the projection of $\vec{V}$ over the $(\hat{i}_3, \hat{i}_2)$-plane. Therefore we can write

$$\Delta v_1\hat{i}_1 = -M(1-\cos\chi)\Big[(v - V\cos\zeta)\hat{i}_v - V\sin\zeta\cos\eta\hat{i}_\theta - V\sin\zeta\sin\eta\hat{i}_\varphi\Big]$$

and

$$\hat{i}_1 = (v - V\cos\zeta)/g\hat{i}_v - V\sin\zeta\cos\eta/g\hat{i}_\theta - V\sin\zeta\sin\eta/g\hat{i}_\varphi. \quad (10)$$

After these transformations we note that $\hat{i}_1$ can be written as $(v - V\cos\theta)/g\hat{i}_v - V\sin\zeta\cos\eta/g\hat{i}_\theta - V\sin\zeta\sin\eta/g\hat{i}_\varphi$ in the laboratory coordinate system. Since the unit vector $\hat{i}_2$ can be taken perpendicular to $\vec{v}$ and $\hat{i}_1$ as

$$\hat{i}_2 = \sin\eta\hat{i}_\theta - \cos\eta\hat{i}_\varphi \quad (11)$$

the unit vector $\hat{i}_3$ must be in the form

$$\hat{i}_3 = \frac{1}{g}\Big[V\sin\zeta\hat{i}_v + (v - V\cos\zeta)\cos\eta\hat{i}_\theta + (v - V\cos\zeta)\sin\eta\hat{i}_\varphi\Big]. \quad (12)$$

Hence, by using (10–12), the velocity change $\Delta\vec{v}$ can be written as

$$\begin{aligned}\Delta\vec{v} = {} & M\left[V\sin\zeta\sin\phi\sin\chi - (1-\cos\chi)(v - V\cos\zeta)\right]\hat{i}_v \\ & + M\left[g\cos\phi\sin\chi\sin\eta - (1-\cos\chi)V\sin\zeta\cos\eta\right. \\ & \left. + (v - V\cos\zeta)\cos\eta\sin\chi\sin\phi\right]\hat{i}_\theta \\ & + M\left[-g\cos\phi\sin\chi\cos\eta - V\sin\zeta\sin\eta(1-\cos\chi)\right. \\ & \left. + (v - V\cos\zeta)\sin\eta\sin\chi\sin\phi\right]\hat{i}_\varphi \end{aligned} \quad (13)$$

which can be used to compute the Fokker-Planck coefficients in the laboratory system.

First it is convenient to compute the following quantities a_1, a_2 and b_2 in the relative coordinate system defined by

$$a_1 = \int_0^\pi \left(\frac{ZZ'e^2}{4\pi\epsilon_0 m'}\right)^2 \frac{\sin\chi}{(1-\cos\chi)} d\chi, \quad (14)$$

$$a_2 = \int_0^\pi \left(\frac{ZZ'e^2}{4\pi\epsilon_0 m'}\right)^2 \sin\chi d\chi, \quad (15)$$

and

$$b_2 = \int_0^\pi \left(\frac{ZZ'e^2}{4\pi\epsilon_0 m'}\right)^2 \frac{\sin^3\chi}{(1-\cos\chi)^2} d\chi. \quad (16)$$

In (14) the integral clearly diverges at the lower χ limit. Particles separated by distances greater than Debye length are shielded from one another and individual particle interactions give way to collective behavior. Therefore, usually, the lower limit χ_{min} of the deflection angle corresponds to an impact parameter equal to the Debye length λ ($\mathrm{cotg}(\chi_{min}/2) = \lambda/p_c$) with p_c the impact parameter [12]. It is easy to verify that in solid state applications the integral (14) does not diverge because of stronger potential screenings; this could be seen if the Brooks and Herring cross-section is used instead of the Coulomb one. In any case, as a first order approximation, the Coulomb cross-section can be used even in solid state problems if the Debye length is substituted to the more appropriate Fermi length λ_F.

If we make the change of variable $u = \mathrm{cotg}(\chi/2)$, from which $du = -d\chi/(1-\cos\chi)$, recalling $\sin\chi = 2u/(1+u^2)$ and $\cos\chi = (u^2-1)/(1+u^2)$, then a_1 can be written as

$$\begin{aligned}a_1 &= \left(\frac{ZZ'e^2}{4\pi\epsilon_0 m'}\right)^2 \int_0^\Lambda \frac{2u}{1+u^2} du \\ &= \left(\frac{Ze^2}{4\pi\epsilon_0 m'}\right)^2 \ln(1+\Lambda^2), \end{aligned} \quad (17)$$

with $\Lambda = \lambda/p_c = \mathrm{cotg}(\chi_{min}/2)$. Let us turn the attention to a_2 given by

$$a_2 = \int_0^\pi \left(\frac{ZZ'e^2(1-\cos\chi)}{16\pi\epsilon_0 m'}\right)^2 \frac{\sin\chi}{(1-\cos\chi)^2} d\chi. \quad (18)$$

After the usual change of variables ($u = \mathrm{cotg}(\chi/2)$) we obtain

$$a_2 = \frac{Z^2Z'^2e^4}{8\pi^2\epsilon_0^2 m'^2} \frac{\Lambda^2}{(\Lambda^2+1)}. \quad (19)$$

Finally in a similar manner we can calculate b_2 as

$$\begin{aligned}b_2 &= \int_0^\pi \frac{Z^2Z'^2e^4\sin\chi^2}{16\pi^2\epsilon_0^2 m'^2} \frac{\sin\chi}{(1-\cos\chi)^2} d\chi \\ &= \frac{Z^2Z'^2e^4M^2}{8\pi^2\epsilon_0^2 m'^2}\left[\ln(1+\Lambda^2) - \frac{\Lambda^2}{(\Lambda^2+1)}\right]. \end{aligned} \quad (20)$$

$$\langle(\Delta v_v)^2\rangle = n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\int_0^{2\pi}d\eta\int_0^{\pi}d\zeta\int_0^{\infty}dV\int_0^{2\pi}d\chi\int_0^{\pi}d\phi V^2 e^{-\frac{m_f V^2}{2kT_f}}$$
$$\times\frac{\left(\frac{ZZ'e^2}{4\pi\epsilon_0 m'}\right)^2}{g^3(1-\cos\chi)^2}M^2\left[V\sin\zeta\sin\phi\sin\chi-(1-\cos\chi)(v-V\cos\zeta)\right]^2\sin\zeta\sin\chi \tag{22}$$

$$\langle(\Delta v_v)^2\rangle = n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\int_0^{\infty}\int_{g_{min}}^{v+V}V^2 e^{-\frac{m_f V^2}{2kT_f}}\frac{1}{g^2}\frac{\pi^2 M^2}{vV}\left[2b_2V^2\sin^2\zeta(g)+4a_2(v-V\cos\zeta)^2(g)\right]dgdV$$
$$= n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\pi^2M^2\left[2b_2\left(\int_0^v I_{1>}(V,v)dV+\int_v^{\infty}I_{1<}(v,V)dV\right)+4a_2\left(\int_0^v I_{2>}(V,v)dV+\int_v^{\infty}I_{2<}(v,V)dV\right)\right] \tag{23}$$

The computation of (2), by using (14–16), is reduced to

$$\langle\Delta\vec{v}\rangle = \hat{i}_v n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\int_0^{2\pi}d\eta\int_0^{\pi}d\zeta\int_0^{\infty}dV$$
$$\times\int_0^{2\pi}d\chi\int_0^{\pi}d\phi V^2 e^{-\frac{m_f V^2}{2kT_f}}\frac{\left(\frac{ZZ'e^2}{4\pi\epsilon_0 m'}\right)^2}{g^3(1-\cos\chi)^2}M$$
$$\times\left[V\sin\zeta\sin\phi\sin\chi-(1-\cos\chi)(v-V\cos\zeta)\right]$$
$$\times\sin\zeta\sin\chi, \tag{21}$$

since all the other components vanish in the integration over ϕ.

Now by using (17–20) we can compute the drift coefficient (2) as

$$\langle\Delta\vec{v}\rangle = \hat{i}_v 2\pi a_1 M n_f\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}V^2$$
$$\times e^{-\frac{m_f V^2}{2kT_f}}(v-V\cos\zeta)\frac{1}{g^3}\sin\zeta d\zeta d\eta dV.$$

Since $g^2 = V^2+v^2-2Vv\cos\zeta$ we can change the variable ζ with g and $gdg = -Vv\sin\zeta d\zeta$. The above equation becomes

$$\langle\Delta\vec{v}\rangle = -\hat{i}_v 4\pi^2 a_1 M n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\int_0^{\infty}\int_{g_{min}}^{v_f+v}\frac{V^2}{vV}$$
$$\times e^{-\frac{m_f V^2}{2kT_f}}\left(v-V\frac{v^2+V^2-g^2}{2vV}\right)\frac{1}{g^2}dgdV,$$

where g_{min} is $V-v$ if $V>v$ or $v-V$ if $v>V$. The integrals in g for both cases can be computed as

$$\int_{V-v}^{V+v}\frac{V}{v}e^{-\frac{m_f V^2}{2kT_f}}\left(v-V\frac{v^2+V^2-g^2}{2vV}\right)\frac{1}{g^2}dg = 0$$

for $V>v$ and

$$\int_{v-V}^{V+v}\frac{V}{v}e^{-\frac{m_f V^2}{2kT_f}}\left(v-V\frac{v^2+V^2-g^2}{2vV}\right)\frac{1}{g^2}dg = \frac{2e^{-\frac{m_f V^2}{2kT_f}}V^2}{v^2}$$

for $v>V$. Therefore we have

$$\langle\Delta\vec{v}\rangle = \hat{i}_v 4\pi^2 a_1 M n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\int_0^{v}\frac{2e^{-\frac{m_f V^2}{2kT_f}}V^2}{v^2}dV$$
$$= 4\pi^2 a_1 M n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\frac{1}{\sqrt{\frac{m_f}{2kT_f}}}\frac{\sqrt{\pi}}{2}$$
$$\times\left[\frac{\mathrm{erf}[\sqrt{x}]-\frac{2}{\sqrt{\pi}}x\exp[-x^2]}{x^2}\right]\hat{i}_v$$

where $x=\sqrt{m_f v^2/2kT_f}$.

Let us turn the attention to the tensor $\langle\Delta v_i\Delta v_j\rangle$. The $\hat{i}_v\hat{i}_v$-component of the diffusion tensor takes the form

see equation (22) above.

After integration in χ and ϕ, and after changing the variable ζ with g we have

see equation (23) above,

where g_{min} is $V-v$ if $V>v$ or $v-V$ if $v>V$. Again the integral can be computed analytically, so that

$$I_{1>}(V,v) = \int_{V-v}^{V+v}\frac{V^2 e^{-\frac{m_f V^2}{2kT_f}}}{vV}V^2$$
$$\times\left(1-\left(\frac{v^2+V^2-g^2}{2vV}\right)^2\right)\frac{1}{g^2}dg = \frac{4}{3}e^{-\frac{m_f V^2}{2kT_f}}V$$

for $V>v$,

$$I_{1<}(v,V) = \int_{v-V}^{V+v}\frac{V^2 e^{-\frac{m_f V^2}{2kT_f}}}{vV}V^2$$
$$\times\left(1-\left(\frac{v^2+V^2-g^2}{2vV}\right)^2\right)\frac{1}{g^2}dg = \frac{4}{3}\frac{e^{-\frac{m_f V^2}{2kT_f}}V^4}{v^3}$$

$$\langle(\Delta v_\theta)^2\rangle = n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} V^2 e^{-\frac{m_f V^2}{2kT_f}}\left(\frac{ZZ'e^2}{4\pi\epsilon_0 m'}\right)^2\frac{g}{g^4(1-\cos\chi)^2}M^2$$
$$\times\left[g\cos\phi\sin\chi\sin\eta-(1-\cos\chi)V\sin\zeta\cos\eta+(v-V\cos\zeta)\sin\eta\sin\chi\sin\phi\right]^2\sin\theta\sin\chi d\theta d\varphi dV d\chi d\phi \quad (24)$$

$$\langle(\Delta v_\theta)^2\rangle = n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\int_0^{\infty}\int_{g_{min}}^{v+V} V^2 e^{-\frac{m_f V^2}{2kT_f}}\frac{1}{g^2}\frac{M^2\pi^2}{vV}\left[g^2 b_2+2a_2V^2\sin^2\zeta+b_2(v-V\cos\zeta)^2\right]dgdV$$
$$= n_f\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{3}{2}}\pi^2\left[b_2\left(\int_0^v I_{3>}(v,V)dV+\int_0^{\infty} I_{3<}(v,V)dV\right)\right.$$
$$\left.+b_2\left(\int_0^v I_{2>}(v,V)dV+\int_0^{\infty} I_{2<}(v,V)dV\right)+2a_2\left(\int_0^v I_{1>}(v,V)dV+\int_0^{\infty} I_{1<}(v,V)dV\right)\right] \quad (25)$$

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for $v > V$,

$$I_{2>}(V,v) = \int_{V-v}^{V+v}\frac{V^2 e^{-\frac{m_f V^2}{2kT_f}}}{vV}\times\left(v-V\left(\frac{v^2+V^2-g^2}{2vV}\right)\right)^2\frac{1}{g^2}dg = \frac{2}{3}e^{-\frac{m_f V^2}{2kT_f}}V$$

for $V > v$ and

$$I_{2<}(v,V)=\int_{v-V}^{V+v}\frac{V^2 e^{-\frac{m_f V^2}{2kT_f}}}{vV}\left(V\left(\frac{v^2+V^2-g^2}{2vV}\right)-v\right)^2\times\frac{1}{g^2}dg = \frac{2}{3}\frac{e^{-\frac{m_f V^2}{2kT_f}}V^2(3v^2-2V^2))}{v^3}$$

for $v > V$. Therefore we have

$$\langle(\Delta v)_v^2\rangle=n_f\pi^2\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{2\pi}M^2\left[2b_2\frac{\mathrm{erf}[x]-x\frac{2}{\sqrt{\pi}}\exp[-x^2]}{x^3}+4a_2\frac{\mathrm{erf}[x](x^2-1)+\frac{2}{\sqrt{\pi}}\exp[-x^2]}{x^3}\right]$$

with $x = \sqrt{m_f/2\pi kT_f}$. The computation of $\langle(\Delta v_\theta)^2\rangle = \langle(\Delta v_\varphi)^2\rangle$ is given by

see equation (24) above.

After integration with the respect to the variables χ, ϕ (24) becomes

see equation (25) above,

where g_{min} is $V - v$ if $V > v$ or $v - V$ if $v > V$. Again we can compute the integral for both cases and obtain

$$I_{3>}(V,v) = \int_{V-v}^{V+v}\frac{V^2 e^{-\frac{m_f V^2}{2kT_f}}}{vV}dg = 2e^{-\frac{m_f V^2}{2kT_f}}V$$

for $V > v$ and

$$I_{3<}(v,V) = \int_{v-V}^{V+v}\frac{V^2 e^{-\frac{m_f V^2}{2kT_f}}}{vV}dg = 2\frac{e^{-\frac{m_f V^2}{2kT_f}}V^2}{v}$$

for $v > V$. Therefore we have

$$\langle(\Delta v)_\theta^2\rangle = n_f\pi^2\left(\frac{m_f}{2\pi kT_f}\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{2\pi}M^2\times\left[2a_2\frac{\mathrm{erf}[x]-x\frac{2}{\sqrt{\pi}}\exp[-x^2]}{x^3}+b_2\frac{\mathrm{erf}[x](x^2-1)+\frac{2}{\sqrt{\pi}}\exp[-x^2]}{x^3}+b_2\frac{\mathrm{erf}[x]}{x}\right]$$

with $x = \sqrt{m_f/2\pi kT_f}$. The other tensor components $\langle(\Delta v_v\Delta v_\theta)\rangle$, $\langle(\Delta v_v\Delta v_\varphi)\rangle$ and $\langle(\Delta v_\theta\Delta v_\varphi)\rangle$ are equal to zero.

Thus the dynamical-friction vector can be written as

$$\langle\Delta\vec{v}\rangle = -\frac{a(v)}{v^2}\hat{i}_v \quad (26)$$

and the diffusion-in-velocity tensor

$$\langle\Delta\vec{v}\Delta\vec{v}\rangle = 2\frac{1}{v}\begin{bmatrix} b(v) & 0 & 0\\ 0 & c(v) & 0\\ 0 & 0 & c(v)\end{bmatrix}, \quad (27)$$

where the coefficients $a(v)$, $b(v)$ and $c(v)$ are given by

$$a(v) = 2\pi a_1 M n_f\left[\mathrm{erf}[x]-\frac{2}{\sqrt{\pi}}xe^{-x^2}\right], \quad (28)$$

$$b(v) = \frac{\pi M^2}{2}n_f\left[b_2\frac{\mathrm{erf}[x]-\frac{2}{\sqrt{\pi}}xe^{-x^2}}{x^2}+2a_2\frac{\mathrm{erf}[x](x^2-1)-\frac{2}{\sqrt{\pi}}xe^{-x^2}}{x^2}\right] \quad (29)$$

and

$$c(v) = \frac{\pi M^2}{4}n_f\left[2a_2\frac{\mathrm{erf}[x]-\frac{2}{\sqrt{\pi}}xe^{-x^2}}{x^2}+b_2\frac{\mathrm{erf}[x](2x^2-1)+\frac{2}{\sqrt{\pi}}xe^{-x^2}}{x^2}\right] \quad (30)$$

respectively ($x = \sqrt{m_f v^2/2kT_f}$).

3 Fokker-Planck equation

3.1 Fokker-Planck equation in spherical coordinates

In order to write the Fokker-Planck equation in the coordinates system (v, θ, φ) the expression of the current (6) and the diffusion equation (1) are needed. The divergence of the tensor $\vec{\nabla} \cdot (\langle \Delta\vec{v}\Delta\vec{v}\rangle f(t,\vec{v}))$ can be written in spherical coordinates by using [13]

$$\vec{\nabla}\cdot\widetilde{a} = \begin{cases} \left(\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 a_{vv}) + \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}(a_{v\theta}\sin\theta) + \frac{1}{v\sin\theta}\frac{\partial}{\partial\varphi}(a_{v\varphi}) - \frac{a_{\theta\theta}+a_{\varphi\varphi}}{v}\right)_v \\ \left(\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 a_{v\theta}) + \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}(a_{\theta\theta}\sin\theta) + \frac{1}{v\sin\theta}\frac{\partial}{\partial\varphi}(a_{\theta\varphi}) - \frac{a_{\theta v}-\mathrm{cotg}\theta a_{\varphi\varphi}}{v}\right)_\theta \\ \left(\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 a_{\varphi v}) + \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}a_{\varphi\theta} + \frac{1}{v\sin\theta}\frac{\partial}{\partial\varphi}(a_{\varphi\varphi}) - \frac{a_{v\varphi}+2\mathrm{cotg}\theta a_{\theta\varphi}}{v}\right)_\varphi \end{cases}$$

where $\widetilde{a}$ is a generic tensor. Then the current $\vec{J}(t,\vec{r},v\vec{\Omega})$ becomes

$$\vec{J}(t,\vec{r},v\vec{\Omega}) = \left(\frac{a(v)}{v^2}f(t,\vec{r},v\widehat{i}_v) + 2\frac{c(v)}{v^2}f(t,\vec{r},v\widehat{i}_v) + \frac{1}{v^2}\frac{\partial}{\partial v}[c(v)vf(t,\vec{r},v\widehat{i}_v)]\right)\widehat{i}_v + \left(\frac{c(v)}{v^2}\frac{\partial}{\partial\theta}f(t,\vec{r},v\widehat{i}_v)\right)\widehat{i}_\theta + \left(\frac{c(v)}{v^2\sin\theta}\frac{\partial}{\partial\varphi}f(t,\vec{r},v\widehat{i}_v)\right)\widehat{i}_\varphi. \quad (31)$$

By taking the divergence of the current (31) we have the Fokker-Planck equation in the following form

$$\frac{\partial}{\partial t}f\left(t,\vec{r},v\widehat{i}_v\right) + v\widehat{i}_v\cdot\frac{\partial}{\partial\vec{r}}f\left(t,\vec{r},v\widehat{i}_v\right) + \langle\Sigma_a\rangle f\left(t,\vec{r},v\widehat{i}_v\right) = \frac{1}{v^2}\frac{\partial}{\partial v}\left[(a(v)+2c(v))f\left(t,\vec{r},v\widehat{i}_v\right)\right] + \frac{1}{v^2}\frac{\partial^2}{\partial v^2}\left[vb(v)f\left(t,\vec{r},v\widehat{i}_v\right)\right] + \frac{c(v)}{v^3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}f\left(t,\vec{r},v\widehat{i}_v\right)\right] + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}f\left(t,\vec{r},v\widehat{i}_v\right)\right] + S\left(t,\vec{r},v\widehat{i}_v\right). \quad (32)$$

In order to obtain the equation in $\mu = \cos(\theta)$ variable we set

$$f(t,\vec{r},v,\mu,\varphi)dvd\mu d\varphi dt = f(t,\vec{r},v\widehat{i}_v)v^2\sin(\theta)dvd\varphi d\theta dt \quad (33)$$

or

$$f(t,\vec{r},v\widehat{i}_v) = -f(t,\vec{r},v,\mu,\varphi)/v^2 \quad (34)$$

so that the Fokker-Planck equation for the unknown $f(t,\vec{r},v,\mu,\varphi)$ becomes

$$\frac{\partial}{\partial t}f(t,\vec{r},v,\mu,\varphi) + v\widehat{i}_v(\mu,\varphi)\cdot\frac{\partial}{\partial\vec{r}}f(t,\vec{r},v,\mu,\varphi) + \langle\Sigma_a\rangle(t,\vec{r},v,\mu,\varphi) = \frac{\partial}{\partial v}\left[\frac{(a(v)-2c(v))}{v^2}f(t,v,\mu,\varphi)\right] + \frac{\partial^2}{\partial v^2}\left[\frac{b(v)}{v}f(t,\vec{r},v,\mu,\varphi)\right] + \frac{c(v)}{v^3}\left(\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}f(t,\vec{r},v,\mu,\varphi)\right] + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\varphi^2}f(t,\vec{r},v,\mu,\varphi)\right) + S(t,\vec{r},v,\mu,\varphi).$$

For the current (31) we have

$$\vec{J}(t,\vec{r},v,\mu,\varphi) = -\left[\left(\frac{a(v)+2c(v)}{v^2}\right)f(t,\vec{r},v,\mu,\varphi) + \frac{\partial}{\partial v}\left(\frac{b(v)}{v}f(t,\vec{r},v,\mu,\varphi)\right)\right]\widehat{i}_v + \left(\frac{c(v)}{v^3}\left[(1-\mu^2)\frac{\partial}{\partial\mu}f(t,\vec{r},v,\mu,\varphi)\right]\right)\widehat{i}_\mu + \frac{c(v)}{v^3}\frac{1}{1-\mu^2}\left(\frac{\partial}{\partial\varphi}f(t,\vec{r},v,\mu,\varphi)\right)\widehat{i}_\varphi. \quad (35)$$

3.2 Fokker-Planck equation and its approximations

So far the complete Fokker-Planck equation for the coulomb cross-section has been obtained. If an initial burst of N test particles are emitted at $t = 0$ with velocity $v = v_0$ the diffusion Fokker-Planck equation takes the form

$$\frac{\partial}{\partial t}f(t,\vec{r},v,\mu,\varphi) + v\widehat{i}_v(\mu,\varphi)\cdot\frac{\partial}{\partial\vec{r}}f(t,\vec{r},v,\mu,\varphi) + \langle\Sigma_a\rangle(t,\vec{r},v,\mu,\varphi) = \frac{\partial}{\partial v}\left[\frac{(a(v)-2c(v))}{v^2}f(t,\vec{r},v,\mu,\varphi)\right] + \frac{\partial^2}{\partial v^2}\left[\frac{b(v)}{v}f(t,\vec{r},v,\mu,\varphi)\right] + \frac{c(v)}{v^3}\left(\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}f(t,\vec{r},v,\mu,\varphi)\right] + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\varphi^2}f(t,\vec{r},v,\mu,\varphi)\right) + S(t,\vec{r},v,\mu,\varphi), \quad (36)$$

$$\frac{\partial}{\partial t} f(t,\vec{r},v,\mu,\varphi) + v\widehat{i}_v(\mu,\varphi)\cdot\frac{\partial}{\partial \vec{r}} f(t,\vec{r},v,\mu,\varphi) + \langle\Sigma_a\rangle v f(t,\vec{r},v,\mu,\varphi) = \frac{\partial}{\partial v}\left[\frac{\alpha'}{v^2} f(t,\vec{r},v,\mu,\varphi)\right] + \frac{\partial^2}{\partial v^2}\left[\frac{\beta}{v} f(t,\vec{r},v,\mu,\varphi)\right] + \frac{\gamma}{v^3}\left(\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu} f(t,\vec{r},v,\mu,\varphi)\right] + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\varphi^2} f(t,\vec{r},v,\mu,\varphi)\right) + S(t,\vec{r},v,\mu,\varphi) \quad (40)$$

$$\frac{\partial}{\partial t} f(t,\vec{r},v,\mu,\varphi) + v\widehat{i}_v\cdot\frac{\partial}{\partial \vec{r}} f(t,\vec{r},v,\mu,\varphi) + \langle\Sigma_a\rangle f(t,\vec{r},v,\mu,\varphi) = \frac{\partial}{\partial v}\left[\frac{\pi a_1}{v^2}(1+\frac{kT_f}{m_f v^2}) f(t,\vec{r},v,\mu,\varphi)\right] + \frac{\partial^2}{\partial v^2}\left[\frac{\pi a_1 kT_f}{m_f v^3} f(t,\vec{r},v,\mu,\varphi)\right] + \frac{\pi a_1}{2v^3}\Big(\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu} f(t,\vec{r},v,\mu,\varphi)\right] + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\varphi^2} f(t,\vec{r},v,\mu,\varphi)\Big) + S(t,v,\mu,\varphi) \quad (41)$$

and must satisfy the following velocity boundary and initial conditions

$$\begin{aligned} f(t=0,\vec{r},v,\mu,\varphi) &= 0 \\ \widehat{i}_\mu\cdot\vec{J}(t,\vec{r},v,\mu=-1,\varphi) &= 0 \\ \widehat{i}_\mu\cdot\vec{J}(t,\vec{r},v,\mu=1,\varphi) &= 0 \\ \widehat{i}_v\cdot\vec{J}(t,\vec{r},v=v_0,\mu,\varphi) &= -N\delta(\mu-1)\delta(t) \\ \widehat{i}_v\cdot\vec{J}(t,\vec{r},v=0,\mu,\varphi) &= 0. \end{aligned} \quad (37)$$

The complete Fokker-Planck equation in (36) is a rather difficult equation to solve and therefore, when only the integral distribution $f(t,\vec{r},v) = \int_0^{2\pi}\int_{-1}^{+1} f(t,\vec{r},v,\mu,\varphi)d\mu d\varphi$ is needed, the corresponding Fokker-Planck diffusion equation can be used. The equation can be obtained by integrating the Fokker-Planck equation with respect to μ and φ and substituting the space transport term by the diffusion term $-\vec{\nabla}\cdot vD\vec{\nabla}f(t,\vec{r},v)$ where D is the diffusion coefficient. This procedure gives the following Fokker-Planck equation

$$\frac{\partial}{\partial t} f(t,\vec{r},v) - \vec{\nabla}\cdot vD\vec{\nabla}f(t,\vec{r},v) + \langle\Sigma_a\rangle f(t,\vec{r},v) = \frac{\partial}{\partial v}\left[\frac{\eta(v)}{v^2} f(t,\vec{r},v)\right] + \frac{\partial^2}{\partial v^2}\left[\frac{\beta(v)}{v} f(t,\vec{r},v)\right] + S(t,\vec{r},v), \quad (38)$$

where

$$\eta(v) = 2\pi a_1 M\left[\mathrm{erf}[x] - \frac{2}{\sqrt{\pi}}xe^{-x^2}\right] - \frac{\pi M^2}{2}\left[4a_2\frac{\mathrm{erf}[x]-\frac{2}{\sqrt{\pi}}xe^{-x^2}}{x^2} + b_2\frac{\mathrm{erf}[x](2x^2-1)-\frac{2}{\sqrt{\pi}}xe^{-x^2}}{x^2}\right] \quad (39)$$

with $x = \sqrt{m_f/2kT_f}v$.

It is clear that in (36) or (38) the stationary solution is not a Maxwell distribution, showing therefore that the approximation does not match the stationary solution of the Boltzmann equation. However the Fokker-Planck equation recovers the Maxwellian distribution in the physical limit of electron-ion interactions. If the field particle temperature T_f is much lower than the slowing down temperature (being the test particle temperature) the field particle distribution can be approximated as a delta function in the form $f_f(\vec{v}) = n_f\delta(\vec{v})$. In this case for $T_f \to 0$ or $x\to\infty$ we have the well-known case in [14] where the coefficient $a(v)$, $b(v)$, $c(v)$, if $\Lambda \gg 1$ is assumed, tend to

$$\alpha = n_f M\Theta, \qquad \beta = \frac{n_f M^2\Theta}{2\ln\Lambda}$$
$$\gamma = \frac{n_f M^2\Theta}{2}\left(1-\frac{1}{2\ln\Lambda}\right)$$

and the Fokker-Planck equation becomes [14]

see equation (40) above,

with $\alpha' = \alpha - 2\gamma$. However in this limit the test particles tend to slow down to zero and the Maxwellian field distribution function is neglected. In order to take into account the field particle distribution we need to consider other terms in the expansion. We can note that the Maxwell distribution is solution of the Fokker-Planck equation only in the limit of $\ln\Lambda \gg 1$ and $M = 1$. Both hypotheses are verified in many physical situations and in particular if electron-ion interactions are considered. In the limit of high $\ln\Lambda$ the coefficient a_2 tends to zero and if $m_f \gg m_t$ then M tends to 1 and b_2 to a_1. The initial charged particles have higher energy than the medium particles and therefore it is natural to consider valid the Fokker Planck equation only in the limit of low T_f, i.e., x tending to ∞. In this case we have

see equation (41) above,

which, for the integrated distribution function $f(t,\vec{r},v)$, becomes

$$\frac{\partial}{\partial t} f(t,\vec{r},v) - \vec{\nabla}\cdot vD\vec{\nabla}f(t,\vec{r},v) + \langle\Sigma_a\rangle f(t,\vec{r},v) = \frac{\partial}{\partial v}\left[\frac{\pi a_1}{v^2}(1+\frac{kT_f}{m_f v^2}) f(t,\vec{r},v,\mu,\varphi)\right] + \frac{\partial^2}{\partial v^2}\left[\frac{\pi a_1 kT_f}{m_f v^3} f(t,\vec{r},v,\mu,\varphi)\right] + S(t,v,\mu,\varphi). \quad (42)$$

It is easy to verify that in absence of the absorption term and infinite medium the stationary solution of (41) and (42) is the Maxwell distribution $f(v) = Cv^2\exp[-m_f v^2/(2kT_f)]$.

4 Solution of the stationary Fokker-Planck equation

Since the stationary solution of (41) is the Maxwell distribution in agreement with the kinetic theory we can use (41) to study the stationary Fokker-Planck equation in presence of the absorption term. We consider an infinite medium and a source that emits a spatially uniform, monochromatic, collimated burst of N test particles per *cm* and *sec* with high velocity $\vec{v}_0$ in the direction $\mu = 1$. Let $\vec{n}$ be the unit vector normal to the velocity sphere S_{v_0} (defined by $S_v = \{\vec{x} : |\vec{x}| = v_0\}$ and directed outwards the spherical surface S_{v_0}. In the three-dimensional velocity space, the existence of this monochromatic source can be expressed by the boundary condition

$$\vec{n} \cdot \vec{J}(v, \mu, \varphi) = -N\delta(v - v_0)\delta(\mu - 1)/2\pi. \tag{43}$$

The absorption $\langle \Sigma_a \rangle(v)$ is considered at low order proportional to $1/E$. Therefore $\langle \Sigma_a \rangle = a/v^2$ with a positive constant [18]. The stationary problem is reduced to the solution of the Fokker-Planck equation

$$\begin{aligned} &-\frac{a}{v} f(t, v, \mu, \varphi)] + \frac{\partial}{\partial v}\left[\frac{\pi a_1}{v^2}(1 + \frac{kT_f}{m_f v^2}) f(t, v, \mu, \varphi)\right] \\ &\qquad + \frac{\partial^2}{\partial v^2}\left[\frac{\pi a_1 kT_f}{m_f v^3} f(t, v, \mu, \varphi)\right] \\ &+ \frac{\pi a_1}{2v^3}\left(\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial}{\partial \mu} f(t, v, \mu, \varphi)\right] + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial \varphi^2} f(t, v, \mu, \varphi)\right) = 0, \end{aligned} \tag{44}$$

with following boundary conditions

$$\begin{aligned} &\widehat{i}_\mu \cdot \vec{J}(t, v, \mu = -1, \varphi) = 0, \qquad \widehat{i}_\mu \cdot \vec{J}(t, v, \mu = 1, \varphi) = 0, \\ &\widehat{i}_v \cdot \vec{J}(t, v = v_0, \mu, \varphi) = -N\delta(\mu - 1)\delta(t)/2\pi, \\ &\widehat{i}_v \cdot \vec{J}(t, v = 0, \mu, \varphi) = 0, \qquad \widehat{i}_\varphi \cdot \vec{J}(t, v, \mu, \varphi = 0) = 0, \\ &\widehat{i}_\varphi \cdot \vec{J}(t, v, \mu, \varphi = 2\pi) = 0 \end{aligned} \tag{45}$$

in the variables (v, μ, φ). If we set

$$f(v, \mu, \varphi) = Y(\mu, \varphi)X(v) \tag{46}$$

we find that $Y(\mu, \varphi) = Y_{n,m}(\mu, \varphi)$ satisfies the spherical harmonic equation

$$\frac{\partial}{\partial \mu}\left[(1 - \mu^2)\frac{\partial}{\partial \mu} Y_{n,m}(\mu, \varphi)\right] + \frac{1}{1 - \mu^2}\frac{\partial^2}{\partial \varphi^2} Y_{n,m}(\mu, \varphi) = - n(n + 1)Y_{n,m}(\mu, \varphi),$$

with the boundary conditions in (45) and $X_n(v)$ satisfies

$$\begin{aligned} &-\frac{a}{v} X_n(v) + \frac{\partial}{\partial v}\left[\frac{1}{v^2}\left(1 + \frac{kT_f}{m_f x^2}\right) X_n(v)\right] \\ &\qquad + \frac{\partial^2}{\partial v^2}\left[\frac{kT_f}{m_f v^3} X_n(v)\right] - n(n + 1)\frac{1}{2v^3} X_n(v) = 0 \end{aligned} \tag{47}$$

with $n = 0, 1, ...$ integer and $-n \le m \le n$. It is convenient to set $z = m_f v^2/2kT_f$ so that (47) becomes

$$zX_n'' + (z - 2)zX_n' + \left[2 - z - \frac{n(n + 1)z}{4} - \frac{az^2}{2}\right] X_n = 0. \tag{48}$$

In order to solve (48) we can set

$$X_n(z) = z^2 \exp\left[-\frac{1}{2}(1 + \sqrt{1 + 2az})\right] \Phi_n(z)$$

and obtain

$$z\Phi'' + (z - 2)\Phi' - \left(1 + \frac{n(n + 1)}{4\sqrt{1 + 2a}}\right) \Phi(z) = 0. \tag{49}$$

The equation (49) is the well-known confluent hypergeometric equation and therefore the solution can be written as

$$\begin{aligned} f(v, \mu, \varphi) = &\sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_{n,m}(\mu, \varphi) \exp\left[-\frac{1}{2}(1 + \sqrt{1 + 2a}\frac{m_f v^2}{2kT_f}\right] \left(\frac{m_f v^2}{2kT_f}\right)^2 \\ &\times \left\{A_n \Phi\left(1 + \frac{n(n + 1)}{4\sqrt{1 + 2a}}, 2; \sqrt{1 + 2a}\frac{m_f v^2}{2kT_f}\right) + B_n U\left(1 + \frac{n(n + 1)}{4\sqrt{1 + 2a}}, 2; \sqrt{1 + 2a}\frac{m_f v^2}{2kT_f}\right)\right\}, \end{aligned} \tag{50}$$

where $\Phi(, ; \cdot)$ and $U(, ; \cdot)$ are the confluent hypergeometric function and the Kummer function respectively. If we compute the current at $v = v_0$ we have

$$\begin{aligned} J_v(v_0) = \vec{J}(v_0, \mu, \varphi) \cdot \widehat{i}_v = \sum_{n=0}^{\infty} J_{vn}(v_0) = \frac{1}{8} \exp\left[-\frac{1}{2}\left(1 + \sqrt{1 + 2a}\right)\frac{m_f v_0^2}{2kT_f}\right] \sum_{n=0}^{\infty} \sum_{m=-n}^{m=n} \left(4\left(2 + \left(1 - \sqrt{1 + 2a}\right)\frac{m_f v_0^2}{2kT_f}\right)\right. \\ \times\left[A_n \Phi\left(1 + \frac{n\,(n + 1)}{4\sqrt{1 + 2a}}, 2; \sqrt{1 + 2a}\frac{m_f v_0^2}{2kT_f}\right) + B_n U\left(1 + \frac{n(n + 1)}{4\sqrt{1 + 2a}}, 2; \sqrt{1 + 2a}\frac{m_f v_0^2}{2kT_f}\right)\right] + \left(4\sqrt{1 + 2a} + n(n + 1)\right)\frac{m_f v_0^2}{2kT_f} \\ \left.\times\left[A_n \Phi\left(2 + \frac{n(n + 1)}{4\sqrt{1 + 2a}}, 3; \sqrt{1 + 2a}\frac{m_f v_0^2}{2kT_f}\right)\right) + 2B_n U\left(2 + \frac{n(n + 1)}{4\sqrt{1 + 2a}}, 3; \sqrt{1 + 2a}\right)\frac{m_f v_0^2}{2kT_f}\right]\right). \end{aligned}$$

The current term in $J_{vn}(v_0)$ obtained by the confluent hypergeometric function ($A_n = 0$)

$$B_n \exp\left[-\frac{1}{2}(1+\sqrt{1+2a})\frac{m_f v_0^2}{2kT_f}\right]\left(\frac{m_f v_0^2}{2kT_f}\right)^2 U\left(1+\frac{n(n+1)}{4\sqrt{1+2a}}, 2; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)$$

diverges for $n \geq 1$ and tends to

$$\frac{1-\sqrt{1+2a}}{2\sqrt{1+2a}} \tag{51}$$

for $n = 0$ when v_0 tends to zero. By multiplying by $Y_{n,m}(\mu,\varphi)$ the boundary condition $J_v(v_0) = N\delta(\mu-1)/2\pi$ and integrating μ and φ over $(-1,1)\times(0,2\pi)$ we obtain

$$A_n \frac{1}{8}\exp\left[-\frac{1}{2}\left(1+\sqrt{1+2a}\right)\frac{m_f v_0^2}{2kT_f}\right]\left\{4\left(2+\left(1-\sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right)\Phi\left(1+\frac{n(n+1)}{4\sqrt{1+2a}}, 2; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right.$$
$$\left.+(4\sqrt{1+2a}+n(n+1))\frac{m_f v_0^2}{2kT_f}\Phi\left(2+\frac{n(n+1)}{4\sqrt{1+2a}}, 3; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right\} = \frac{N}{2\pi^2 a_1}\frac{2n+1}{2}$$

and $B_n = 0$ for $n > 0$. For $n = 0$ we have

$$A_0\left\{\frac{1}{8}e^{-\frac{1}{2}(1+\sqrt{1+2a})\frac{m_f v_0^2}{2kT_f}}\left[4\left(2+\left(1-\sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right)\Phi\left(1, 2; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right.\right.$$
$$\left.\left.+(4\sqrt{1+2a})\frac{m_f v_0^2}{2kT_f}\Phi\left(2, 3; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right]\right\} + B_0\left\{\frac{1}{8}e^{-\frac{1}{2}(1+\sqrt{1+2a})\frac{m_f v_0^2}{2kT_f}}\left[4\left(2+\left(1-\sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right)\right.\right.$$
$$\left.\left.\times U\left(1, 2; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)+2\left(4\sqrt{1+2a}\right)\frac{m_f v_0^2}{2kT_f}U\left(2, 3; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right]\right\} = \frac{N}{4\pi^2 a_1}$$

$$A_0 + B_0\frac{1-\sqrt{1+2a}}{2\sqrt{1+2a}} = 0.$$

Therefore the constants can be written as

$$A_0 = \frac{N\sqrt{1+2a}(\sqrt{1+2a}-1)\ \exp\left[\frac{m_f v_0^2}{4kT_f}(1+\sqrt{1+2a})\right]}{2\pi^2 a_1 a\left(\exp\left[\sqrt{1+2b}\frac{m_f v_0^2}{2kT_f}\right]-1\right)}$$

$$A_n = \frac{2N\exp\left[\frac{1}{2}(1+\sqrt{1+2a})\frac{m_f v_0^2}{2kT_f}\right](2n+1)}{\pi^2 a_1\left(4(2+\left(1-\sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right)\Phi\left(1+\frac{n(n+1)}{4\sqrt{1+2a}}, 2; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)}$$
$$\times\frac{1}{\frac{m_f v_0^2}{2kT_f}\left(4\sqrt{1+2a}+n(n+1)\right)\Phi\left(2+\frac{n(n+1)}{4\sqrt{1+2a}}, 3; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\Big)} \qquad n > 0$$

$$B_0 = \frac{N(1+2a)\exp\left[\frac{v_0^2}{2}(1+\sqrt{1+2a})\right]}{\pi^2 a_1 a(\exp\left[\sqrt{1+2b}v_0^2\right]-1)} \qquad B_n = 0 \qquad n > 0$$

The solution of the beam problem is

$$f(v,\mu,\varphi) = \frac{N}{4\pi^2 a_1}\exp\left[-\frac{1}{2}(1+\sqrt{1+2a}\frac{m_f(v^2-v_0^2)}{2kT_f}\right]$$
$$\times\left[\frac{\left(\frac{m_f v^2}{2kT_f}\right)^2}{a(\exp\left[\sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right]-1)}\left(\sqrt{1+2a}(\sqrt{1+2a}-1)\Phi\left(1, 2; \sqrt{1+2a}\frac{m_f v^2}{2kT_f}\right)+2(1+2a)U\left(1, 2; \sqrt{1+2a}\frac{m_f v^2}{2kT_f}\right)\right)\right.$$
$$\left.+8\left(\frac{m_f v^2}{2kT_f}\right)\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{2}Y_{n,m}(\mu,\varphi)C_n^{-1}\Phi\left(1+\frac{n(n+1)}{4\sqrt{1+2a}}, 2; \sqrt{1+2a}\frac{m_f v^2}{2kT_f}\right)\right]. \tag{52}$$

with $C_n = \left(4\left(2+\left(1-\sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right)\Phi\left(1+\frac{n(n+1)}{4\sqrt{1+2a}}, 2; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right.$
$$\left.+\left(4\sqrt{1+2a}+n(n+1)\right)\frac{m_f v_0^2}{2kT_f}\Phi\left(2+\frac{n(n+1)}{4\sqrt{1+2a}}, 3; \sqrt{1+2a}\frac{m_f v_0^2}{2kT_f}\right)\right)$$

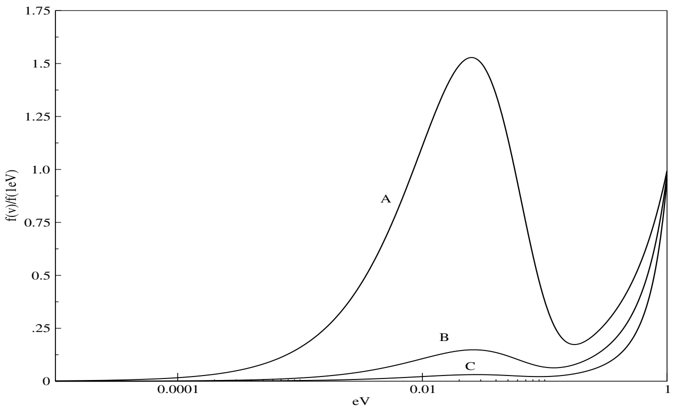

Fig. 1. Electron distribution function $f(v)$ as a function of energy $E = m_t v^2/2$ for different values of $a = 0.01$ (A), 0.05 (B) and 0.1 cm s^{-2} (C) for a 1 eV source.

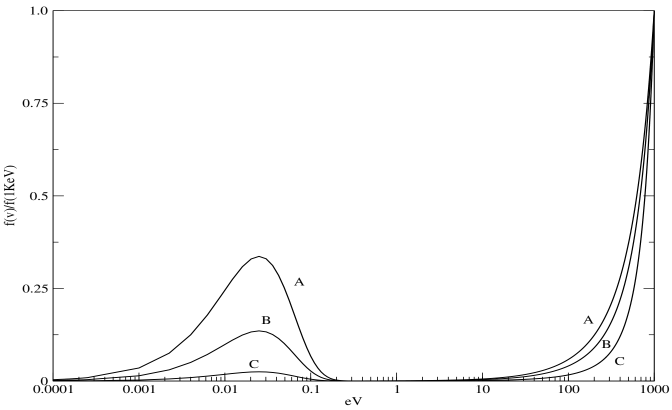

Fig. 3. Electron distribution function $f(v)$ as a function of energy $E = m_t v^2/2$ for different values of $a = 1 \times 10^{-5}$ (A), 5×10^{-5} (B) and 1×10^{-4} cm s^{-2} (C) for a 1 keV source.

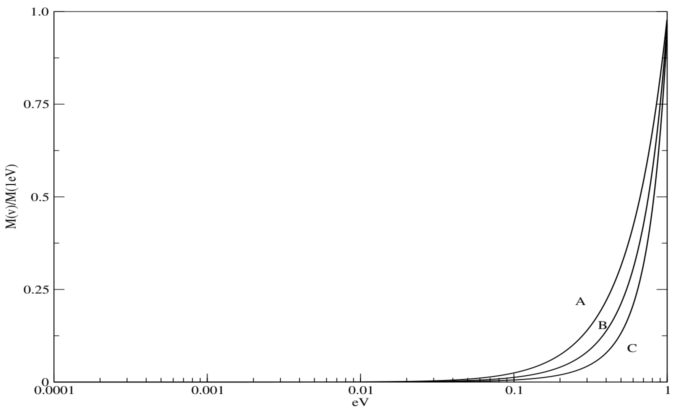

Fig. 2. Average electron direction $M(v)$ as a function of energy $E = m_t v^2/2$ for different values of $a = 0.1$ (A), 0.05 (B) and 0.01 cm s^{-2} (C) for a 1 eV source.

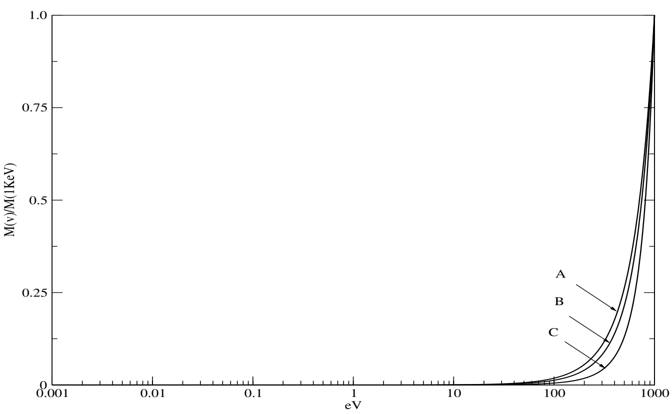

Fig. 4. Average electron direction $M(v)$ as a function of energy $E = m_t v^2/2$ for different values of $a = 1 \times 10^{-4}$ (A), 5×10^{-5} (B) and 1×10^{-5} cm s^{-2} (C) for a 1 keV source.

and the corresponding integrated distribution function is

$$f(v) = \frac{N}{4\pi^2 a_1} \exp\left[-\frac{1}{2}\left(1 + \sqrt{1+2a\frac{m_f(v^2 - v_0^2)}{2kT_f}}\right)\right] \times \left[\frac{\left(\frac{m_f v^2}{2kT_f}\right)^2}{a\left(\exp\left[\sqrt{1+2a\frac{m_f v_0^2}{2kT_f}}\right] - 1\right)} \times \left(\sqrt{1+2a}(\sqrt{1+2a} - 1)\Phi\left(1, 2; \sqrt{1+2a}\frac{m_f v^2}{2kT_f}\right) + 2(1+2a)U\left(1, 2; \sqrt{1+2a}\frac{m_f v^2}{2kT_f}\right)\right)\right]. \quad (53)$$

As a first example a very low energy electron source is considered. Since the thermal energy is about 0.025 eV a source of 1 eV is sufficient to see the behavior of the solution when the particles slow down from thermal energy. In Figure 1 the integral distribution function $f(v)$ is plotted as a function of energy $E = m_t v^2/2$ for different values of the absorption coefficient a: $a = 0.01$ (A), 0.05 (B) and 0.1 cm s^{-2} (C). For the typical values of a the solution is almost a Maxwellian distribution since the electrons cannot be absorbed efficiently. The corresponding average angle $M(v) = \int_{-1}^{1} \mu f(v, \mu, \varphi) d\mu d\varphi$ is plotted in Figure 2 as a function of the electron energy E. Since (52) is written as a sum of spherical harmonics the integration is particularly straightforward. The beam particle is emitted with sharp anisotropy and after the quick slowing down the particle distribution function becomes completely isotropic. In Figures 3 and 4 the same results are shown for a 1 keV electron source. In Figure 3 we see that the slowing down spectrum and the equilibrium spectrum are separated and are active at completely different energy range. A typical slowing down spectrum is recovered in the high energy range and a Maxwellian-like behavior is located in the thermal region. In Figures 3 and 4 the electron distribution functions $f(v)$ and the corresponding average angle function $M(v)$ are plotted for $a = 1 \times 10^{-5}$ (A), 5×10^{-5} (B) and 1×10^{-4} cm s^{-2} (C). Finally in Figures 5 and 6 the solutions for a 100 keV electron source are computed. Again the Maxwellian part of the distribution function $f(v)$ in Figure 5 tends to disappear the more the absorption coefficient becomes important. The cases $a = 2.5 \times 10^{-7}$ (A), 5×10^{-7} (B) and 1×10^{-6} cm s^{-2} (C) are presented. The behavior of the average angle $M(v)$,

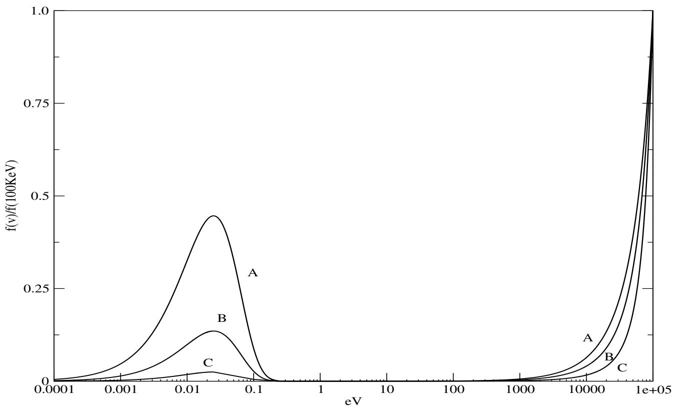

Fig. 5. Electron distribution function $f(v)$ as a function of energy $E = m_t v^2/2$ for different values of $a = 2.5 \times 10^{-7}$ (A), 5×10^{-7} (B) and 1×10^{-6} $\mathrm{cm\,s^{-2}}$ (C) for a 100 keV source.

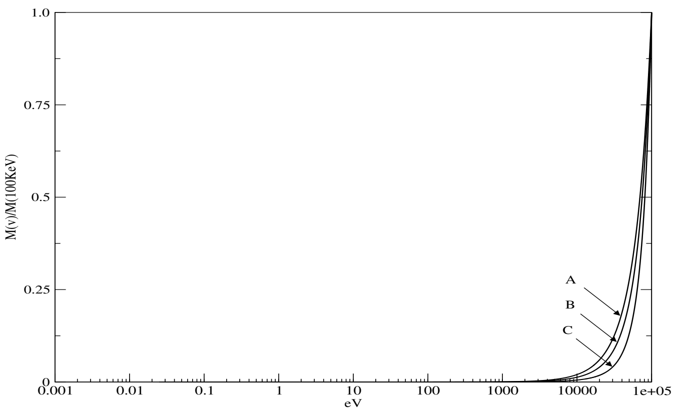

Fig. 6. Average electron direction $M(v)$ as a function of energy $E = m_t v^2/2$ for different values of $a = 1 \times 10^{-6}$ (A), 5×10^{-7} (B) and 2.5×10^{-7} $\mathrm{cm\,s^{-2}}$ (C) for a 100 keV source.

shown in Figure 6, is similar to those generated by the previous source energy.

5 Conclusions

In this paper the charged particle slowing down, modeled by the Fokker-Planck equation taking into account the field particle distribution at low energy in the Maxwellian form, has been investigated. In the electron-ion interaction approximation the solution tends to take the form of a Maxwell distribution in the limit of no absorption and the form of a classical slowing down spectrum for high absorption rates. High absorption rates imply also slower isotropization of an initially completely anisotropic beam, generating thus a kind of energy-angle correlation. In general for a medium with low absorption rate the Maxwell distribution of the thermalized particles may be of relevance in many applications where an accurate spectrum need to be known [20].

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